Kuwait University | **Math 102** | December 14, 2008

Dept. Maths. & Comp. Sci. | Second Midterm | Duration: 90 minutes

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Calculators and mobile phones are not allowed.

Answer all of the following questions.

1. Evaluate the following limits. (2 points each)

(a) $\displaystyle\lim_{x\to 0} \frac{\cos 7x - \cos 3x}{2x^2}$

(b) $\displaystyle\lim_{x\to 0^+} (1+\sin 3x)^{\frac{1}{\tan^{-1}(2x)}}$

2. Evaluate the following integrals. (3.5 points each)

(a) $\displaystyle\int \frac{x^3+4x}{x^2-4}\,dx$

(b) $\displaystyle\int \frac{\sin^{\frac{5}{2}} x}{\sec^3 x}\,dx$

(c) $\displaystyle\int x^2\, 2^x\, dx$

(d) $\displaystyle\int \frac{x}{(x^2-4x+8)^{\frac{3}{2}}}\,dx$

(e) $\displaystyle\int \frac{x^{\frac{1}{2}}-1}{x(x^{\frac{1}{6}}+x^{\frac{1}{2}})}\,dx$

(f) $\displaystyle\int \tanh^5 x\, \operatorname{sech} x\, dx$

1. (a) $\displaystyle\lim_{x\to 0}\frac{\cos 7x-\cos 3x}{2x^2}\overset{\text{L'H}}{=}\lim_{x\to 0}\frac{-7\sin 7x+3\sin 3x}{4x}\overset{\text{L'H}}{=}\lim_{x\to 0}\frac{-49\cos 7x+9\cos 3x}{4}=-10$

(b) $\displaystyle\lim_{x\to 0^+}\frac{\ln(1+\sin 3x)}{\tan^{-1}(2x)}\overset{\text{L'H}}{=}\lim_{x\to 0^+}\frac{\frac{3\cos 3x}{1+\sin 3x}}{\frac{2}{1+4x^2}}=\frac{3}{2}$ therefore $\displaystyle\lim_{x\to 0^+}(1+\sin 3x)^{\frac{1}{\tan^{-1}(2x)}}=e^{\frac{3}{2}}$

2. (a) $\displaystyle\int\frac{x^3+4x}{x^2-4}\,dx=\int\left(x+\frac{8x}{(x-2)(x+2)}\right)dx=\frac{x^2}{2}+\int\left(\frac{4}{x-2}+\frac{4}{x+2}\right)dx$

$\displaystyle=\frac{x^2}{2}+4\left(\ln|x-2|+\ln|x+2|\right)+C$

(b) Let $u=\sin x$. Then $\displaystyle\int\sin^{\frac{5}{2}}x\cos^3 x\,dx=\int u^{\frac{5}{2}}(1-u^2)du=\int\left(u^{\frac{5}{2}}-u^{\frac{9}{2}}\right)du$

$\displaystyle=\frac{2}{7}\sin^{\frac{7}{2}}x-\frac{2}{11}\sin^{\frac{11}{2}}x+C$

(c) We use integration by parts twice with $dv=2^x\,dx$:

$\displaystyle\int x^2 2^x\,dx=\frac{x^2 2^x}{\ln 2}-\frac{2}{\ln 2}\int x2^x\,dx=\frac{x^2 2^x}{\ln 2}-\frac{2}{\ln 2}\left(\frac{x2^x}{\ln 2}-\frac{1}{\ln 2}\int 2^x\,dx\right)$

$\displaystyle=\frac{x^2 2^x}{\ln 2}-\frac{2}{\ln 2}\left(\frac{x2^x}{\ln 2}-\frac{2^x}{(\ln 2)^2}\right)+C$

(d) $\displaystyle I=\int\frac{x\,dx}{(x^2-4x+8)^{\frac{3}{2}}}=\int\frac{x\,dx}{((x-2)^2+4)^{\frac{3}{2}}}$. Let $x-2=2\tan\theta$. Then

$\displaystyle I=\int\frac{2+2\tan\theta}{8\sec^3\theta}2\sec^2\theta\,d\theta=\int\frac{1+\tan\theta}{2\sec\theta}\,d\theta=\frac{1}{2}\int(\cos\theta+\sin\theta)\,d\theta$

$\displaystyle=\frac{1}{2}(\sin\theta-\cos\theta)+C=\frac{1}{2}\left(\frac{x-2}{\sqrt{x^2-4x+8}}-\frac{2}{\sqrt{x^2-4x+8}}\right)+C=\frac{x-4}{2\sqrt{x^2-4x+8}}+C$

(e) Let $x=u^6$. Then $\displaystyle\int\frac{x^{\frac{1}{2}}-1}{x(x^{\frac{1}{6}}+x^{\frac{1}{2}})}\,dx=\int\frac{u^3-1}{u^6(u+u^3)}6u^5\,du=6\int\frac{u^3-1}{u^2(1+u^2)}\,du$

$\displaystyle=6\int\left(-\frac{1}{u^2}+\frac{u+1}{1+u^2}\right)du=6\left(\frac{1}{u}+\frac{1}{2}\ln\left(1+u^2\right)+\tan^{-1}u\right)+C$

$\displaystyle=\frac{6}{\sqrt[6]{x}}+3\ln\left(1+\sqrt[3]{x}\right)+6\tan^{-1}\sqrt[6]{x}+C$

(f) $\displaystyle I=\int\tanh^5 x\,\operatorname{sech}x\,dx=\int\left(1-\operatorname{sech}^2x\right)^2\tanh x\,\operatorname{sech}x\,dx$. Let $u=\operatorname{sech}x$. Then

$\displaystyle I=-\int\left(1-u^2\right)^2\,du=-\int\left(1-2u^2+u^4\right)du=-\operatorname{sech}x+\frac{2}{3}\operatorname{sech}^3x-\frac{1}{5}\operatorname{sech}^5x+C$